ON THE APPLICATION OF THE LIAPUNOV METHOD TO STABILITY PROBLEMS

(O PRIMENENII METODA LIAPUNOVA K ZADACHEM USTOICHIVOSTI)

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1. Certain definitions.* Consider the linear system of differential equations of disturbed motion

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij}(t)\, x_j \qquad (i = 1, \ldots, n) \tag{1.1}$$

where the $p_{ij}(t)$ are given, continuous, bounded functions of time.

Let the n^2 numbers $p_{ij}(i,\ j = l,\ \ldots,\ n)$ be the coordinates of a point of the n^2-dimensional space P. The curve having in the space P the parametric equations $p_{ij}(t)$ will be called the coefficient line of the system of linear differential equations (1.1).

Let $V(t:\ x_1,\ x_2,\ \ldots,\ x_n)$ be a sign definite function [1]. By (1.1), the time derivative

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \sum_{j=1}^{n} p_{ij}(t)\, x_j + \frac{\partial V}{\partial t} \tag{1.2}$$

is a known function of time and of the coefficients of the system (1.1). The concept of the region $L(V)$ will be introduced in the following manner.

The set of points of the n^2-dimensional space of coefficients P, to each of which for every $t > t_0$ the derivative dV/dt is a function of

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constant sign, opposite to the sign of V, will be called the region $L(V)$, corresponding to a given sign definite function V.

Obviously, it follows from Liapunov's stability theorem [1] that an undisturbed motion is stable, if there exists a sign definite function to which there corresponds a region $L(V)$ containing the coefficient line of the system of equations of the disturbed motion. The function V will then be the Liapunov function of the system under consideration.

2. Liapunov functions in the form of quadratic forms with constant coefficients. Let

$$V = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad (\alpha_{ij} = \alpha_{ji})$$

be a positive definite form with constant coefficients. By (1.1), its derivative with respect to time will obviously also be a quadratic form:

$$\frac{dV}{dt} = \sum_{ij=1}^{n} a_{ij}(t) x_i x_j \tag{2.1}$$

where the a_{ij} are functions of time, determined by the formulas

$$a_{ij}(t) = \sum_{s=1}^{n} (\alpha_{is} p_{sj}(t) + \alpha_{js} p_{si}(t)) \qquad (i;\ j = 1, \ldots, n) \tag{2.2}$$

$$a_{ij} = a_{ji}$$

In the present case, the conditions determining the region $L(V)$ will be the Sylvester's known conditions for the quadratic form to be negative:

$$(-1)^s \det_s |a_{ij}| \geqslant 0 \qquad (s = 1, \ldots, n,\ i = 1, \ldots, n) \tag{2.3}$$

A set of points of the n^2-dimensional space P will be called a Routh-Hurwitz region, if inside it the conditions of the Routh-Hurwitz theorem for the polynomial

$$\Delta(\lambda) = \det|p_{ij} - \delta_{ij}\lambda| = 0, \qquad \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

are fulfilled.

The following proposition holds true: the region $L(V)$ corresponding to any sign definite quadratic form is contained inside the Routh-Hurwitz region.

In fact, the proposition is true, since otherwise one could construct a system of linear differential equations with constant coefficients which does not satisfy the conditions of Routh-Hurwitz, but which is stable on the strength of Liapunov's stability theorem. The contradiction proves the statement. It is likewise readily verified that to every sign definite quadratic form V there corresponds a non-empty region $L(V)$. In fact, putting $p_{ij} = -a_{ij}$, one finds

$$\frac{dV}{dt} = 2\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\alpha_{ij}x_j\Big)\Big(\sum_{j=1}^{n}p_{ij}x_j\Big) = -2\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\alpha_{ij}x_j\Big)^2$$

i.e. a point of the space P with the coordinates $p_{ij} = -a_{ij}$ belongs to the region $L(V)$, and hence, by the continuity of the region $L(V)$, so does a certain part of its neighborhood.

Further, it will be proved that the region $L(V)$ is bounded by a surface of an n^2-dimensional cone.

In fact, if the point with coordinates $p_{ij} = p_{ij}^{\circ}$ belongs to the region $L(V)$, then any point of the half ray from the origin of coordinates through the point $\{p_{ij}^{\circ}\}$ likewise belongs to the region $L(V)$. This half ray is determined in parametric form by the equations

$$p_{ij} = kp_{ij}^{\circ} \qquad (i,\ j = 1, \ldots, n)$$

where k is a positive parameter. Obviously, by (2.1) and (2.2),

$$\frac{dV}{dt}(x_s,\ p_{kl}) = k\frac{dV}{dt}(x_s,\ p_{kl}^{\circ})$$

i.e. for every $k > 0$, a point $\{p_{ij}\}$ belongs to the region $L(V)$.

3. The equation and properties of the surface, bounding the region $L(V)$. Let V be a positive definite quadratic form with constant coefficients and $p_{ij} = p_{ij}(\tau)$ $(i,\ j = 1,\ 2,\ \ldots,\ n)$ some line in space P. Further, let for $\tau = \tau_0$ the point

$$p_{ij}^{\circ} = p_{ij}(\tau_0)$$

belong to the region $L(V)$.

The following proposition will be proved: if for $\tau = \tau_0$ $(p_{ij}^{\circ} = p_{ij}(\tau_0))$ the coefficients of the quadratic form dV/dt satisfy Sylvester's conditions (2.3), then for a continuous change of the parameter τ, from the value τ_0 only the last of Sylvester's conditions (2.3) may at first be violated.

The truth of this proposition follows from the fact that, if among the series of principal diagonal minors $D_1, \ldots, D_n$ of the discriminant of the quadratic form the minor $D_k (n > k > 1)$ vanishes, the values of the the minors D_{k-1} and D_{k+1} must be of opposite signs [3].

Consequently, points on the boundary of the region $L(V)$ satisfy the equation

$$D_n = \det_n \|a_{ij}\| = 0 \qquad (3.1)$$

where the a_{ij} are determined by (2.2).

The determinant D_n will be considered as a function of the coefficients of the k-th equation of the system $p_{k1}, p_{k2}, \ldots, p_{kn}$.

On the basis of (2.2), one may write

$$a_{ij} = \alpha_{ik}p_{kj} + \alpha_{jk}p_{ki} + A_{ij}^{(k)} \tag{3.2}$$

$$A_{ij}^{(k)} = \sum_{s \neq k}^{n} (\alpha_{is}p_{sj} + \alpha_{js}p_{si}) \tag{3.3}$$

Representing the determinant (3.1) in the form of the sum of determinants, the elements of which will be terms of the elements of the determinant (3.1), it will be noted that the determinants, in which two or more columns consist of elements being first or second terms in the elements of the determinant (3.1), are zero as they have two or more columns proportional to each other.

Thus, the determinant (3.1) may be represented in the form of the sum of determinants of the following four types:

(1) Determinants containing two columns with elements having the p_{ks} ($s = 1, 2, \ldots, n$) as multipliers; for this purpose, if one of the columns consists of elements which are the first terms of the corresponding elements of the determinants (3.1), then the second column must consist of the second terms of the corresponding elements of the determinants (3.1).

(2) Determinants which contain one column consisting of the first terms of the corresponding elements (3.2).

(3) Determinants which contain one column, consisting of the second terms of the corresponding elements (3.2).

(4) The determinant det $A_{ij}^{(k)}$ . .

Denote the sums of the determinants of the above form by S_1, S_2, S_3, S_4 respectively and their sum by

$$D_n = S_1 + S_2 + S_3 + S_4 \tag{3.4}$$

It is easily seen that

$$S_1 = \sum_{s;\ r\ s \neq r}^{n} \det{}_n \| (1 - \delta_{js} - \delta_{jr}) A_{ij}^{(k)} + \delta_{js}\alpha_{ik}p_{kj} + \delta_{jk}\alpha_{jk}p_{ki} | \tag{3.5}$$

$$S_2 = \sum_{s=1}^{n} \det{}_n \| (1 - \delta_{js}) A_{ij}^{(k)} + \delta_{js}\alpha_{ik}p_{kj} \| \tag{3.6}$$

$$S_3 = \sum_{s=1}^{n} \det_n \| (1 - \delta_{js}) A_{ij}^{(k)} + \delta_{js} \alpha_{jk} p_{ki} \| \tag{3.7}$$

$$S_4 = \det_n | A_{ij}^{(k)} \| \tag{3.8}$$

where

$$\delta_{sr} = \begin{cases} 1 & \text{for } s = r \\ 0 & \text{for } s \neq r \end{cases} \tag{3.9}$$

Taking out the common factors of the s-th and r-th columns, by adding the determinants of the sum S_1 and the common factor of the s-th column, by adding the determinants of the sums S_2 and S_3, one obtains

$$S_1 = \sum_{s=1}^{n} p_{ks} \sum_{r=1, r \neq s}^{n} \alpha_{rk} \det_n | (1 + \delta_{js} + \delta_{jr}) A_{ij}^{(k)} + \delta_{js} \alpha_{ik} + \delta_{jr} p_{ki} |$$

$$S_2 = \sum_{s=1}^{n} p_{ks} \det_n | (1 - \delta_{js}) A_{ij}^{(k)} + \delta_{js} \alpha_{ik} |$$

The sum S_1 may be written in the form of the $(n + 2)$-nd order determinai

$$S_1 = - \begin{vmatrix} 0 & 0 & p_{k1} \cdots p_{kn} \\ 0 & 0 & \alpha_{k1} \cdots \alpha_{kn} \\ p_{k1} & \alpha_{k1} & \\ \cdots & \cdots & |A_{ij}^{(k)}| \\ p_{kn} & \alpha_{kn} & \end{vmatrix} \tag{3.10}$$

and the sums S_2 and S_3 in the form the $(n + 1)$-st order determinants

$$S_2 = - \begin{vmatrix} 0 & p_{k1} \cdots p_{kn} \\ \alpha_{k1} & \\ \vdots & |A_{ij}^{(k)}| \\ \alpha_{kn} & \end{vmatrix} \qquad S_3 = - \begin{vmatrix} 0 & \alpha_{k1} \cdots \alpha_{kn} \\ p_{k1} & \\ \vdots & |A_{ij}^{(k)}| \\ p_{kn} & \end{vmatrix} \tag{3.11}$$

Obviously $S_2 \equiv S_3$. Replacing in the determinant S_1 the off-diagonal zeros by (-1), one obtains for the determinant $D_n = S_1 + S_2 + S_3 + S_4$ the expression

$$D_n = - \begin{vmatrix} 0 & -1 & p_{k1} \cdots p_{kn} \\ -1 & 0 & \alpha_{k1} \cdots \alpha_{kn} \\ p_{k1} & \alpha_{k1} & \\ p_{kn} & \alpha_{kn} & |A_{ij}^{(k)}| \end{vmatrix} \tag{3.12}$$

It will now be verified that $D_n = 0$ represents a surface in the n-dimensional space of the coefficients $p_{k1}, \ldots, p_{kn}$ of the k-th equation. Obviously, the equation $D_n = 0$ is the equation of a second order surface.

The following theorem will now be proved:

Theorem. The region $L(V)$ corresponding to the positive definite form

$$V = \sum_{i;j-1}^{n} \alpha_{ij} x_i x_j$$

in the n-dimensional space of coefficients of any of the equations of disturbed motion is an elliptic paraboloid, with vector components of the asymptotic direction being proportional to the values of the corresponding coefficients of the quadratic form V.

Proof: The determinant D_n may be represented as a general second degree form

$$D_n = \sum_{ij=1}^{n} H_{ij} p_{ki} p_{kj} + 2 \sum_{i=1}^{n} H_i p_{ki} + H \qquad (3.13)$$

where, obviously,

$$\sum_{ij=1}^{n} H_{ij} p_{ki} p_{kj} \equiv S_1, \qquad \sum_{i=1}^{n} H_i p_{ki} \equiv S_2 \equiv S_3, \qquad H = S_4 \qquad (3.14)$$

In order that $D_n = 0$ will be a paraboloid, the discrimant of the quadratic form must vanish:

$$\sum_{ij-1}^{N} H_{ij} p_{ki} p_{kj} \quad \text{or} \quad \det \| H_{ij} \| = 0 \qquad (3.15)$$

Obviously, the coefficient H_{ij} is obtained from the determinant

$$S = \begin{vmatrix} 0 & \alpha_{k1} \ldots \alpha_{kn} \\ \alpha_{k1} & \\ \alpha_{kn} & | A_{ij}^{(k)} | \end{vmatrix} \qquad (3.16)$$

by striking the $(i + 1)$st row and the $(j + 1)$st column

$$H_{ij} = (-1)^{i+j} S_{ij} \qquad (3.17)$$

where S_{ij} is the minor corresponding to the element $A_{ij}^{(k)}$ of the determinant (3.16). It will now be proved that

$$\sum_{j-1}^{n} H_{ij} \alpha_{kj} = 0 \qquad (i = 1, \ldots, n) \qquad (3.18)$$

Indeed, this follows from the fact that the left hand side of each of these equalities represent determinants with two equal columns, while not all of the a_{kj} $(j = 1, \ldots, n)$ vanish on the strength of the sign definite property of the quadratic form V, whence the following condition must be satisfied

$$\det \| H_{ij} \| = 0 \qquad (3.19)$$

i.e. the region $L(V)$ is a paraboloid. The components $k_1 k_2 \ldots k_n$ of the asymptotic direction of the second order surface satisfy the known equation

$$\sum_{ij=1}^{n} H_{ij} k_i k_j = 0 \tag{3.20}$$

Obviously the quantities $\alpha_{k1} \alpha_{k2} \ldots \alpha_{kn}$ satisfy this equation.

In fact, by (3.18),

$$\sum_{ij=1}^{n} H_{ij} \alpha_{ki} \alpha_{kj} = \sum_{i=1}^{n} \alpha_{ki} \sum_{j=1}^{n} H_{ij} \alpha_{kj} = 0$$

Thus, $D_n = 0$ is the equation of a paraboloid with the components of the asymptotic direction proportional to the quantities $\alpha_{k1} \alpha_{k2} \ldots \alpha_{kn}$, i.e. the theorem is proved.

The transformation which was used to reduce the determinant D_n to the form (3.12) may be applied to the determinant $\det_n \| A_{ij}^{(k)} \|$. In fact, the elements $A_{ij}^{(k)}$ may be represented in the form

$$A_{ij}^{(k)} = \alpha_{il} p_{lj} + \alpha_{jl} p_{li} + A_{ij}^{(kl)}, \quad A_{ij}^{(kl)} = \sum_{\substack{s \neq k \\ s \neq l}}^{n} (\alpha_{is} p_{sj} + \alpha_{js} p_{si}) \tag{3.21}$$

Hence, the elements $A_{ij}^{(k)}$ have assumed a form analogous to (3.12), and the determinant D_n may be written

$$D_n = \begin{vmatrix} 0 & -1 & 0 & 0 & p_{k1} & \cdots & p_{kn} \\ -1 & 0 & 0 & 0 & \alpha_{k1} & \cdots & \alpha_{kn} \\ 0 & 0 & 0 & -1 & p_{l1} & \cdots & p_{ln} \\ 0 & 0 & -1 & 0 & \alpha_{l1} & \cdots & \alpha_{ln} \\ p_{k1} & \alpha_{k1} & p_{l1} & \alpha_{l1} & A_{11}^{(kl)} & \cdots & A_{m}^{(kl)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{kn} & \alpha_{kn} & p_{ln} & \alpha_{ln} & A_{n1}^{(kl)} & \cdots & A_{nn}^{(kl)} \end{vmatrix} \tag{3.22}$$

Similar transformations may be performed while the superscripts k, l do not exhaust all the values from 1 to n. For this all the $A_{ij}^{(k, l, \ldots)}$ vanish and the determinant obtains finally the form

$$D_n = (-1)^n \times \tag{3.23}$$

$$\times \begin{vmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 & p_{11} & p_{12} & \dots & p_{1n} \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 & p_{21} & p_{22} & \dots & p_{2n} \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & p_{n1} & p_{n2} & \dots & p_{nn} \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \\ p_{11} & \alpha_{11} & p_{21} & \alpha_{21} & \dots & p_{n1} & \alpha_{n1} & 0 & 0 & \dots & 0 \\ p_{12} & \alpha_{12} & p_{22} & \alpha_{22} & \dots & p_{n2} & \alpha_{n2} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m} & \alpha_{1n} & p_{2n} & \alpha_{2n} & \dots & p_{nn} & \alpha_{nn} & 0 & 0 & \dots & 0 \end{vmatrix}$$

The coordinates of the points of contact of the paraboloid (3.12) with the coordinate planes (in the n-dimensional space of the coefficients of the k-th equation $p_{k1}p_{k2}\dots p_{kn}$) will now be determined. Let in the determinant D_n

$$p_{ki} = p_{ki}^{(s)} = -\frac{A_{si}^{(k)}}{\alpha_{ks}} + \nu_s \alpha_{ki} \tag{3.24}$$

For this purpose, the elements of the first row of the determinant D_n of (3.12), beginning with the second, will be equal to linear combinations of the corresponding elements of the $(S + 2)$th and the second rows.

Using the arbitrariness of the factor ν_s, it may be determined from the condition

$$\frac{p_{ks}^{(s)}}{\alpha_{ks}} + \nu_s = 0 \tag{3.25}$$

but, by (3.24)

$$p_{ks}^{(s)} = -\frac{A_{ss}^{(k)}}{\alpha_{ks}} + \nu_s \alpha_{ks} \tag{3.26}$$

Substituting in (3.25), one finds

$$\frac{A_{ss}^{(k)}}{\alpha_{ks}^2} - 2\nu_s = 0, \qquad \text{or} \qquad \nu_s = \frac{A_{ss}^{(k)}}{2\alpha_{ks}^2} \tag{3.27}$$

Substitution of this expression for ν_s in (3.24) gives

$$p_{ki}^{(s)} = \frac{1}{\alpha_{ks}}\left(\frac{A_{ss}^{(k)}}{2}\,\frac{\alpha_{ki}}{\alpha_{ks}} - A_{si}^{(k)}\right) \qquad (i = 1, \dots, n) \tag{3.28}$$

For these values of $p_{k1}, \dots, p_{kn}$ [corresponding to arbitrarily fixed values $s (s = 1, \dots, n)$], the determinant D_n vanishes.

Thus, n points $P_1, \ldots, P_n$ are obtained which belong to the boundary of the region $L(V)$. The values of the derivatives $\partial D_n/\partial p_{ki}$ at the points $p_1, \ldots, p_n$ will now be evaluated. One has

$$\frac{1}{2}\frac{\partial D_n}{\partial p_{ki}} = (-1)^i \begin{vmatrix} -1 & p_{k1} & p_{k2} & \cdots & p_{kn} \\ 0 & \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kn} \\ \alpha_{k1} & A_{11}^{(k)} & A_{12}^{(k)} & \cdots & A_{1n}^{(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{ki-1} & A_{i-1,1} & A_{i-1,2} & \cdots & A_{i-1,n} \\ \alpha_{ki+1} & A_{i+1,1} & A_{i+1,2} & \cdots & A_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{kn} & A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} \tag{3.29}$$

Obviously, by (3.25), all partial derivatives vanish at the point P_s except for $\partial D_n/\partial p_{ks}$, i.e.

$$\left.\frac{\partial D_n}{\partial p_{ki}}\right|_{p_{ki}=p_{ki}^{(s)}} \begin{cases} = 0 & (i \neq s) \\ \neq 0 & (i = s) \end{cases} \qquad (s = 1, \ldots, n)$$

This means that the paraboloid $D_n = 0$ touches at the point P_s the plane $p_{ks} = p_{ks}^{(s)}$, parallel to the coordinate plane $p_{ks} = 0$ at the point with the coordinates $p_{k1}^{(s)}, p_{k2}^{(s)}, \ldots, p_{kn}^{s}$, determined by the equalities (3.28).

4. Case of a single n-th order equation. The n-th order equation

$$y^{(n)} + A_1(t)\, y^{(n-1)} + A_2(t)\, y^{(n-2)} + \ldots + A_n(t)\, y = 0 \tag{4.1}$$

may be written in the form of the system

$$\begin{gathered} \dot{x}_1 = p_{11}(t)\, x_1 + p_{12}(t)\, x_2 + \ldots + p_{1n}(t)\, x_n \qquad (p_{1s}(t) = -A_s(t)) \\ \dot{x}_2 = x_1, \ldots, \quad \dot{x}_n = x_{n-1} \qquad (x_s = y^{(n-s)}) \end{gathered} \tag{4.2}$$

In this case, considering the region $L(V)$ as a region of the n-dimensional space of the coefficients $p_{11}p_{12}\ldots p_{1n}$, one has

$$A_{ij}^{(1)} = \sum_{s=2}^{n} (\alpha_{is}p_{sj} + \alpha_{js}p_{si}) = \begin{cases} \alpha_{ij+1} + \alpha_{ji+1} & (i < n,\ j < n) \\ \alpha_{ij+1} & (i = n,\ j < n) \\ \alpha_{ji+1} & (i < n,\ j = n) \\ 0 & (i = n,\ j = n) \end{cases} \tag{4.3}$$

and the equation of the boundary of $L(V)$ has the form

$$D_n = -\begin{vmatrix} 0 & -1 & p_{11} \cdots p_{1n} \\ -1 & 0 & \alpha_{11} \quad \alpha_{1n} \\ p_{11} & \alpha_{11} & \\ p_{1n} & \alpha_{1n} & \boxed{\;|A_{ij}^{(1)}|\;} \end{vmatrix} \tag{4.4}$$

Equation (4.4) represents in the space of the coefficients $p_{11}, \ldots, p_{1n}$ an elliptic paraboloid with the components of the vector of the asymptotic direction, proportional to the quantities $a_{11}, \ldots, a_{1n}$ and touching the coordinate planes at the points with coordinates [cf. (3.28), (4.3)]

$$p_{1i}^{(s)} = \frac{1}{\alpha_{1s}}\left(\alpha_{s,s+1}\frac{\alpha_{1i}}{\alpha_{1s}} - \alpha_{s,\,i+1} - \alpha_{is+1}\right) \tag{4.5}$$

The formulas (4.5) show that the region $L(V)$ in the present case touches the coordinate plane $p_{1n} = 0$ at the point

$$p_{11}^{(n)} = -\frac{\alpha_{2n}}{\alpha_{1n}}, \quad p_{12}^{(n)} = -\frac{\alpha_{3n}}{\alpha_{1n}}, \ldots, p_{1n-1}^{(n)} = -\frac{\alpha_{nn}}{\alpha_{1n}}, \quad p_{1n}^{(n)} = 0 \tag{4.6}$$

The derived simple dependence of the coordinates of the points of contact of the region $L(V)$ with the coordinate planes of the space of coefficients $p_{11}, \ldots, p_{1n}$ may in concrete cases be essentially simplified by finding the Liapunov function in the form of a quadratic form with constant coefficients.

In the present case, it is obvious that not every positive definite quadratic form V corresponds to a non-empty region $L(V)$ in the space of coefficients $p_{11}\ldots p_{1n}$. The supplementary conditions will now be studied, which must be satisfied by the coefficients a_{ij} of the quadratic form V, so that so that the region $L(V)$ will exist.

For the quadratic form dV/dt Sylvester's determinants will obviously be

$$D_r = -\begin{vmatrix} 0 & -1 & p_{11} \cdots p_{1r} \\ -1 & 0 & \alpha_{11} \cdots \alpha_{1r} \\ p_{11} & \alpha_{11} & \\ \cdots & \cdots & |A_{ij}^{(1)}|_r \\ \cdots & \cdots & \\ p_{1r} & \alpha_{1r} & \end{vmatrix} \qquad (r = 1, \ldots, n) \tag{4.7}$$

where the $|A_{ij}|_r$ are the principal diagonal minors of r-th order of the determinant $|A_{ij}^{(1)}|$, and the $A_{ij}^{(1)}$ are determined by the expressions (4.3).

It is sufficient for the existence of the region $L(V)$, if there exists one point at which Sylvester's inequality

$$(-1)^r D_r > 0 \qquad (r = 1, \ldots, n)$$

is fulfilled. Consider the value of Sylvester's determinant at the point of contact of the surface $D_n = 0$ with the plane $p_{1n} = 0$. The coordinates of this point are determined by the expressions (4.6).

If the region $L(V)$ exists, then the inequalities

$$(-1)^r D_r|_{p_{1i}=p_{1i}^{(n)}} > 0 \qquad (r=1,\ldots,n-1), \qquad (-1)^n \frac{\partial D_n}{\partial p_{1n}}\bigg|_{p_{1i}=p_{1i}^{(n)}} < 0 \tag{4.8}$$

must be fulfilled at this point.

The last inequality must hold on the strength of the fact that the region $L(V)$ belongs to the region of Routh-Hurwitz for which $p_{1n}<0$.

Substituting the values $p_{1i}^{(n)}$ from (4.5) into (4.8), one obtains, after elementary transformations, the known conditions:

$$(-1)^{r+1}\begin{vmatrix} 0 & \alpha_{1n} & \alpha_{2n} \cdots \alpha_{r+1,1} \\ \alpha_{1n} & 0 & \alpha_{11} \cdots \alpha_{1r} \\ \alpha_{2n} & \alpha_{11} & \\ \cdots & & |A_{ij}^{(1)}|_r \\ \alpha_{r+1,n} & \alpha_{1r} & \end{vmatrix} > 0 \qquad (r=1,\ldots,n-1) \tag{4.9}$$

$$(-1)^n \frac{1}{\alpha_{1n}}\begin{vmatrix} 0 & \alpha_{11} \cdots \alpha_{1n} \\ \alpha_{11} & \\ \alpha_{1n} & |A_{ij}^{(1)}| \end{vmatrix} > 0 \tag{4.10}$$

Comparing the last inequality (4.8) with the inequality (4.9) (for $r = n - 1$), it is readily verified that the inequality (4.8) may be replaced by the simple condition $a_{1n} > 0$.

Since the region $L(V)$ belongs to the region, determined by the inequalities of Routh-Hurwitz from which there follow the inequalities $p_{1i} < 0$, one has on the basis of (4.6)

$$\alpha_{in} > 0 \qquad (i=1,\ldots n) \tag{4.11}$$

In addition, since $a_{11}, \ldots, a_{1n}$ are the components of a vector, parallel to the axis of the paraboloid and lying in the region where $p_{1i} < 0$ $(i = 1, 2, \ldots, n)$, the quantities $a_{11}, \ldots, a_{1n}$ must all have the same sign.

By the condition of positive definiteness $a_{11} > 0$. Hence, one must have

$$\alpha_{1i} > 0 \qquad (i=1,\ldots,n) \tag{4.12}$$

Thus, the additional conditions have been derived which must be satisfied by the coefficients of the positive definite form

$$V = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \tag{4.13}$$

so that a non-empty region $L(V)$ will exist in the n-dimensional coefficient space:

$$(-1)^{r+1}\begin{vmatrix} 0 & \alpha_{11}\ \dots\ \alpha_{1r} & \dots\ \alpha_{1n} \\ \alpha_{11} & & \alpha_{2n} \\ \dots & |A_{ij}^{(1)}|_r & \dots\dots \\ \alpha_{r1} & & \alpha_{r+1,\,n} \\ \alpha_{n1} & \alpha_{n2}\ \dots\ \alpha_{n,r+1} & \dots\ 0 \end{vmatrix} > 0 \qquad (r = 1, \dots, n-1) \qquad (4.14)$$

$$\alpha_{1i} > 0, \qquad \alpha_{in} > 0 \qquad (i = 1, \dots, n) \qquad (4.15)$$

As an example a problem will be considered which is of interest in automatic control theory.

Let the disturbed motion of the system satisfy the n-th order differential equation

$$y^{(n)} + A_1^\circ y^{(n-1)} + A_2^\circ y^{(n-2)} + \dots + A_{n-1}^\circ y' + A_n^\circ(t)\, y = 0 \qquad (4.16)$$

where A_i° = const. $(i = 1, \dots, n-1)$, $A_n(t) > 0$ for every value $t > 0$ vanishes for $t > 0$.

The constants $A_1^\circ, \dots, A_{n-1}^\circ$ and the function $A_n(t)$ will be assumed to satisfy the conditions of Routh-Hurwitz.

If the quadratic form (4.13) with constant coefficients is the Liapunov function for the system under consideration, the region $L(V)$ must touch the plane $p_{1n} = 0$ at the point with the coordinates $p_{11} = -A_1^\circ p_{12} = -A_2^\circ$, $p_{1n-1} = -A^\circ_{n-1}$. Hence, on the basis of (4.6), the coefficients $a_{1n} a_{2n} \dots a_{nn}$ are related to the $A_1^\circ, \dots, A^\circ_{n-1}$ by the equalities

$$\frac{\alpha_{2n}}{\alpha_{1n}} = A_1^\circ, \qquad \frac{\alpha_{3n}}{\alpha_{1n}} = A_2^\circ, \dots, \frac{\alpha_{nn}}{\alpha_{1n}} = A^\circ_{n-1} \qquad (4.17)$$

Without restricting the generality, one may let $a_{1n} = 1$ (condition (4.14) and then one finds

$$\alpha_{1n} = 1, \qquad \alpha_{2n} = A_1^\circ, \qquad \alpha_{3n} = A_2^\circ, \dots, \alpha_{nn} = A^\circ_{n-1} \qquad (4.18)$$

Replacing in the equations (3.12), which determine the boundary of the region $L(V)$, the quantities $p_{11} p_{12} \dots p_{1n}$ by the quantities $-A_1^\circ\ -A_2^\circ, \dots, -A^\circ_{n-1}\ -A_n$ and substituting the obtained values of $a_{1n}, a_{2n}, \dots, a_{nn}$ one arrives at

$$D_n = -\begin{vmatrix} 0 & 1 & A_1^\circ & A_2^\circ & \dots & A_n \\ 1 & 0 & \alpha_{11} & \alpha_{12} & \dots & 1 \\ A_1^\circ & \alpha_{11} & \alpha_{12} + \alpha_{12} & \alpha_{13} + \alpha_{22} & \dots & A_1^\circ \\ A_2^\circ & \alpha_{12} & \alpha_{22} + \alpha_{13} & \alpha_{23} + \alpha_{23} & \dots & A_2^\circ \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_n & 1 & A_1^\circ & A_2^\circ & \dots & 0 \end{vmatrix} \qquad (4.19)$$

By (4.6), the value $A_n = 0$ is a root of the quadratic equation $D_n(A_n) = 0$. The second root $A_n(\{a_{ij}\})$ $(i, j = 1, \ldots, n-1)$ will be a rational function of the remaining undetermined coefficients a_{ij} of the quadratic form V. These coefficients may always be determined from the system of equations

$$\frac{\partial \bar{A}_n}{\partial \alpha_{ij}} = 0 \qquad (i;\ j = 1, \ldots, n-1) \tag{4.20}$$

Thus, sufficient conditions for the stability of the system under consideration will be

$$\bar{A}_n > A_n(t) > 0$$

In the case of the second-order equation

$$y'' + A_1^\circ y' + A_2(t)\, y = 0 \tag{4.21}$$

where $A_1^\circ > 0$, one has by (4.18) the following values for the coefficient of the Liapunov function

$$\alpha_{12} = 1, \qquad \alpha_{22} = A_1^\circ$$

For this purpose, one has for A_2 the expression (4.19):

$$\bar{A}_2 = 4\,\frac{\alpha_{11}A_1^\circ - 1}{\alpha_{11}^2} \quad \text{or} \quad \bar{A}_2 = A_1^{\circ 2} \tag{4.22}$$

since it follows from the condition $\partial A_2/\partial a_{11} = 0$ that $a_{11} = 2/A_1^\circ$.

Thus, if the function $A_2(t)$ satisfies the condition

$$A_1^{\circ 2} > A_2(t) > 0 \tag{4.23}$$

the undisturbed motion is stable and the quadratic form with constant coefficients

$$V = \frac{2}{A_1^\circ}\, y'^2 + 2y'y + A_1^\circ y^2 \tag{4.24}$$

will be the Liapunov function. In cases where $n > 2$, the solution of the system (4.20) may be obtained by known approximate methods.

5. The stability of one class of non-stationary motions. Let the system of differential equations of the disturbed motion have the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij}(t)\, x_j \qquad (i = 1, \ldots, n) \tag{5.1}$$

where

$$p_{ij}(t) = p_{ij}^\circ + t^{m_1} p_{ij}^{(1)} + \ldots + t^{m} p_{ij}^{(k)} \tag{5.2}$$

$$0 < m_1 < m_2 < \ldots < m_k \text{ — constants}$$

$$p_{ij}^{(s)} \qquad (i;\ j = 1, \ldots, n;\ s = 1, \ldots, k) \text{ — constants}$$

The following new independent variable will be introduced:

$$\tau = \frac{1}{m_k + 1}\, t^{m_k + 1} \tag{5.3}$$

The system of equations of disturbed motion (5.1) then takes the form

$$\frac{dx_i}{d\tau} = \sum_{j=1}^{n} q_{ij}(\tau)\, x_j \qquad (i = 1, \ldots, n) \tag{5.4}$$

where

$$q_{ij}(\tau) = \sum_{s=0}^{k} p_{ij}^{(s)} [(m_k + 1)\tau]^{-\frac{m_k + m_s}{m_k+1}} \qquad (i;\ j = 1, \ldots, n), \qquad m_0 = 0$$

Obviously as $\tau \to \infty$ the coefficients $q_{ij}(\tau)$ tend to definite limits

$$\lim_{\tau\to\infty} q_{ij}(\tau) = p_{ij}^{(k)} \qquad (i,\ j = 1, \ldots, n)$$

and therefore, by the theorem of Chetaev [2], the stability of the obvious solution of the system (5.4), and consequently of the system (5.1), follows from the stability of the obvious solution of the limiting system

$$\frac{dx_i}{d\tau} = \sum_{j=1}^{n} p_{ij}^{(k)} x_j \tag{5.5}$$

From this follows the validity of the following proposition.

Theorem. The trivial solution of the system of linear differential equations of disturbed motion (5.1) with coefficients of the form (5.2) is asymptotically stable if the roots of the characteristic equation of the limiting system (5.5)

$$\det \| p_{ij}^{(k)} - \delta_{ij}\lambda \| = 0$$

satisfy the condition Re el $\lambda_i < \epsilon$, where ϵ is an arbitrarily small fixed negative number.

6. Stability of non-linear systems. The above results may also be applied to the study of the stability of non-linear systems. Let the system of equations of disturbed motion have the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \varphi_{ij}(t, x_1, \ldots, x_n)\, x_j \qquad (i = 1, \ldots, n) \tag{6.1}$$

where $\phi_{ij}(t;\ x_1,\ \ldots,\ x_n)$ are for $t > 0$ continuous and bounded functions in any finite neighborhood of the origin of coordinates. Let

$$V(x_1, \ldots, x_n) = \sum_{i;\ j=1}^{n} \alpha_{ij} x_i x_j \qquad (\alpha_{ij} = \alpha_{ji} = \text{const})$$

be a positive definite quadratic form, the derivative of which on the strength of the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \varphi_{ij}(t, 0, \ldots, 0)\, x_j \qquad (i = 1, \ldots, n)$$

is a negative definite quadratic form. Obviously, then, the quadratic form V will be likewise the Liapunov function for the system (6.1). The results of Sections 1-3 above may be used to estimate the regions of the initial disturbances $x_1^0, \ldots, x_n^0$, to which correspond the solutions $x_1(t), \ldots, x_n(t)$ of the system (6.1), satisfying the condition $x_i(t) \to 0$ for $t \to \infty$. In fact, replacing in the conditions $(-1)^n \det_n \| a_{ij} \| > 0$ the coefficients p_{ij} by the functions $\phi_{ij}(t;\ x_1, \ldots, x_n)$, one obtains in the variables $t;\ x_1, \ldots, x_n$ the sufficient condition

$$(-1)^n \det_n \| a_{ij}(t;\ x_1, \ldots, x_n) \| > 0$$

where

$$a_{ij}(t;\ x_1, \ldots, x_n) = \sum_{s=1}^{n} (\alpha_{is}\varphi_{sj}(t;\ x_1, \ldots, x_n) + \alpha_{js}\varphi_{si}(t;\ x_1, \ldots, x_n))$$

It it is satisfied, then the function

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \sum_{j=1}^{n} \varphi_{ij}(t;\ x_1, \ldots, x_n)\, x_j$$

is negative definite. If $c(t) = \inf V(x_1, \ldots, x_n)$ on the surface $\det_n \| a_{ij}(t;\ x_1, \ldots, x_n) \| = 0$ and if $c_0 = \inf c(t)$ for $t > 0$, then the property $x_i(t) \to 0$ for $t \to \infty$ $(i = 1, \ldots, n)$ applies for the condition $V(x_1^0, \ldots, x_n^0) < c_0$ where $x_i^0 = x_i(0)(i = 1, \ldots, n)$ (the number $c_0 > 0$ exists on the strength of the assumption that dV/dt by (6.2) is negative definite).

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